Name: ______________________

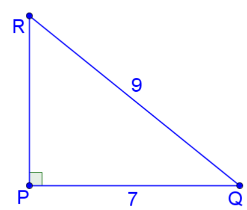
SM3 8.2 Law of Sines

Review: Solving a triangle means finding all of the side and angle measures of the triangle.

Solving right triangles has required the Pythagorean Theorem, $a^2 + b^2 = c^2$, trigonometric functions, $\sin(\,)$, $\cos(\,)$, and $\tan(\,)$, as well as their inverse functions $\arcsin(\,)$, $\arccos(\,)$, and $\arctan(\,)$. Our calculator is typically in "degrees" mode.

Review Example: Given that $m\angle P = 90°, r = 7, p = 9$, solve ΔPQR.

Step 1) Sketch a reasonable representation of the triangle to help decide which pieces of information played each role in the triangle:



Step 2) If you're given two sides, find the third side with the Pythagorean Theorem. If you're given two angles, find the third angle by subtraction.

We're given two sides, so it's Pythagorean Theorem time!

$$a^2 + b^2 = c^2$$
$$q^2 + 7^2 = 9^2$$
$$q^2 + 49 = 81$$
$$q^2 = 32$$
$$q \approx 5.7$$

Step 3) If a side is still missing, use a trig evaluation. If an angle is still missing, use an inverse trig evaluation.

We're still missing angles, so we'll need an inverse trig evaluation!

$$\cos Q = \frac{7}{9}$$
$$m\angle Q = \arccos\left(\frac{7}{9}\right)$$
$$m\angle Q \approx 38.9°$$

Step 4) If a side is still missing, use another trig evaluation. If an angle is still missing, use subtraction.

We're missing an angle, so we'll need to subtract!

$$m\angle P + m\angle Q + m\angle R = 180°$$
$$90° + 38.9° + m\angle R \approx 180°$$
$$m\angle R \approx 51.1°$$

Step 5) Write your solution as the set of values that you solved for.

$$\boxed{q \approx 5.7, m\angle Q \approx 38.9°, m\angle R \approx 51.1°}$$

Note: Impossible cases to solve exist (e.g., the hypotenuse is shorter than a leg, or sum of the legs is smaller than the hypotenuse, etc.).

The time has come to open up your ability to solve any triangle, not just right triangles. In order to solve a triangle, you'll need to find 3 measurements of any combination of sides and angles.

The solution of a triangle (Latin: *solutio triangulorum*) is the historical term for solving the main trigonometric problem of finding the characteristics of a triangle (angles and lengths of sides), when some of these are known. Depending on the measurements you have, you can determine how many solutions you will have, if you even have any at all.

What happens if we know the value of …

… all 3 sides (SSS)

When the sides' lengths are fixed, the angles that hold the sides in position are also fixed. As a result, there will be only 1 solution to the triangle. Begin with the Law of Cosines.

… all 3 angles (AAA)

While the angle measures are fixed, because the triangle could be dilated to a different size, we'll have infinitely many solutions to the triangle. We don't expect you to find these solutions.

… two angles and one side (AAS or ASA)

Because the angles of a triangle always sum to $180°$ and we know two of the angles, we also know the third angle using subtraction. Since we have the length of one side, our triangle is locked into a certain size. This gives us only 1 solution. Begin with the Law of Sines.

… two sides and the angle between them (SAS)

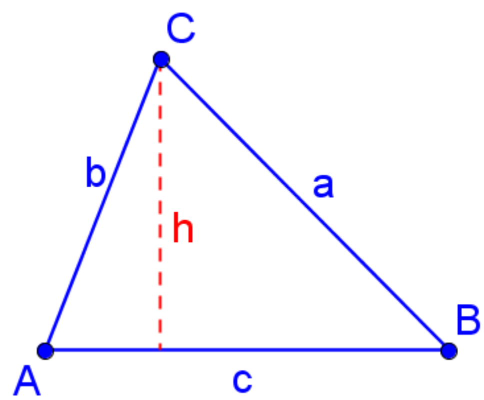
The angle aims the two sides, and they have fixed lengths. There will only be one way to connect the last side of the triangle. Hence, there will be only 1 solution. Begin with the Law of Cosines.

…two sides and an angle that is not between them (SSA)

Depending on the values, a variety of scenarios can take place. We'll need to explore this case together and discover how to proceed.

Law of Sines

Given ΔABC, it is possible to construct the height of the triangle from C to c. Call this h.



$\sin A = \frac{h}{b}$	Definition of sin( ).
$h = b \sin A$	Multiplication
$\sin B = \frac{h}{a}$	Definition of sin( ).
$h = a \sin B$	Multiplication
$b \sin A = a \sin B$	Substitution
$\frac{\sin A}{a} = \frac{\sin B}{b}$	Division

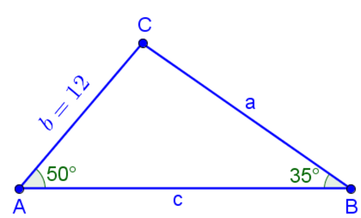
By constructing a height from B to b, we could similarly prove that $\frac{\sin A}{a} = \frac{\sin C}{c}$

By constructing a height from A to a, we could similarly prove that $\frac{\sin B}{b} = \frac{\sin C}{c}$

Law of Sines: Given ΔABC, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Example: Solve ΔABC given that $\text{m}\angle A = 50°$, $\text{m}\angle B = 35°$, and $b = 12$.

Step 1) Sketch the triangle and determine which type of information was given.



It appears we've been given AAS information. We'll proceed with use of the Law of Sines to solve.

Step 2) We only use two ratios of the Law of Sines at one time. Since we know $\text{m}\angle B$ and b, we'll use the ratio that contains both of those terms. The other known information we have is $\text{m}\angle A$, so we'll use the ratio that contains it as well.

$\frac{\sin A}{a} = \frac{\sin B}{b}$	Law of Sines
$\frac{\sin 50°}{a} = \frac{\sin 35°}{12}$	Substitution
$a = \frac{12 \sin 50°}{\sin 35°}$	Solve for a
$a \approx 16.0$	Use a calc, round to nearest tenth.

Step 3) We can find the missing angle by subtraction, and then use the Law of Sines again to find the last missing length. $\text{m}\angle C = 95°$, $c \approx 20.8$

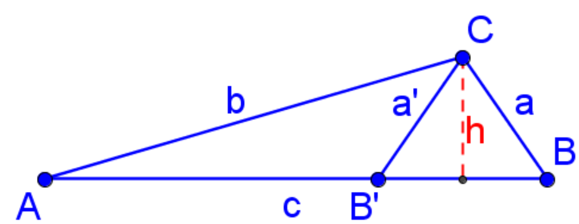
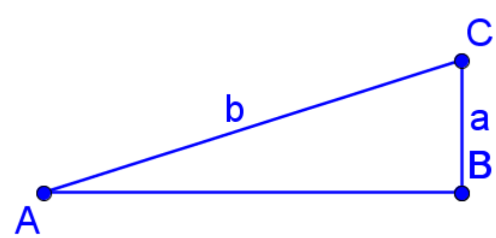
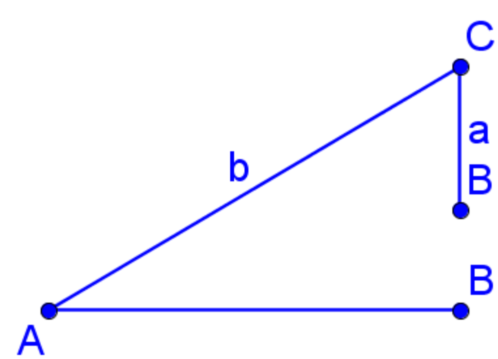
$a \approx 16.0$, $\text{m}\angle C = 95°$, and $c \approx 20.8$

The Ambiguous Case

Given two sides, a and b, and an angle of a triangle not between them, $\angle A$, we have a SSA case. Let's explore the different numbers of solutions the triangle will have, depending on the relationships of the given measurements.

Known measures	$0° < m\angle A < 90°$	$\text{m}\angle A = 90°$	$90° < m\angle A < 180°$
$a = b$			
$a > b$			
$a < b$			

Most of these fields tell us that moving forward with the Laws will get us to the only answer. Only one field sticks out as particularly interesting. We call this the ambiguous case because it isn't clear whether there are 0, 1, or 2 solutions. Let's examine the properties of that case.

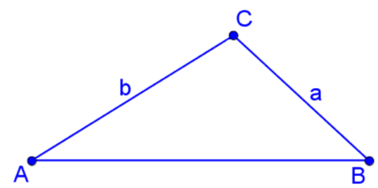
<u>*The Ambiguous Case*</u>:	SSA case wherein $\angle A$ is acute, the side across from $\angle A$, called a is smaller than the other given side, b.
The height, h, of the triangle can be found using the trig identity: $\sin A = \frac{h}{b} \longrightarrow h = b \sin A$. h has to be smaller than a. Therefore, $a > b \sin A$ in any 2 solution model.	2 solutions 
The height, h, of the triangle can be found using the trig identity $\sin A = \frac{h}{b} \longrightarrow h = b \sin A$. $a = h$ as a is the height of the right triangle. Therefore, $a = b \sin A$ in any 1 solution model.	1 solution 
The height, h, if the triangle could be formed is found using the trig identity $\sin A = \frac{h}{b} \longrightarrow$ $h = b \sin A$. $\angle B$ can't be formed because a is not long enough to reach the base of the triangle. This implies that $a < h$. Therefore, $a < b \sin A$ in any 0 solution model.	0 solutions 

<u>The Ambiguous Case</u>: SSA case wherein $\angle A$ is acute, the side across from $\angle A$, called a is smaller than the other given side, b.

$a > b \sin A$	2 solutions
$a = b \sin A$	1 solution
$a < b \sin A$	0 solutions

<u>Example</u>: Solve ΔABC given that $a = 7$, $\text{m}\angle A = 32°$, and $b = 9$

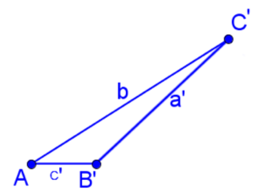
Two sides and an angle (SSA) could mean trouble. Maybe this is an ambiguous case model?



$b \sin A \approx 4.77$ and $7 > 4.77$

Therefore, $a > b \sin A$

This is an ambiguous case model.



Sketch and solve both triangles; the right answer includes both sets of solutions!

Start by finding $\text{m}\angle B$ in the first triangle using the Law of Sines.

$$\frac{\sin 32°}{7} = \frac{\sin B}{9} \longrightarrow \sin B = \frac{9 \sin 32°}{7}$$

$$\text{m}\angle B \approx$$

Use subtraction from 180° to find $\text{m}\angle C$.

$$\text{m}\angle C = 180° - \text{m}\angle A - \text{m}\angle B$$

$$\text{m}\angle C \approx$$

Use the Law of Sines again to find c.

$$\frac{\sin 32°}{7} = \frac{\sin 105.1°}{c} \longrightarrow c = \frac{7 \sin 105.1°}{\sin 32°}$$

$$c \approx 12.8$$

Because $\Delta CBB'$ is isoceles, the base angles are the same. So $\text{m}\angle B'$ in the second case is supplemental to the base angles, namely $\angle B$.

$$\text{m}\angle B' = 180° - \text{m}\angle B$$

$$\text{m}\angle B' =$$

Use subtraction from 180° to find $\text{m}\angle C'$.

$$\text{m}\angle C' = 180° - \text{m}\angle A - \text{m}\angle B$$

$$\text{m}\angle C' \approx$$

Use the Law of Sines to find c'

$$\frac{\sin 32°}{7} = \frac{\sin 10.9°}{c'} \longrightarrow c' = \frac{7 \sin 10.9°}{\sin 32°}$$

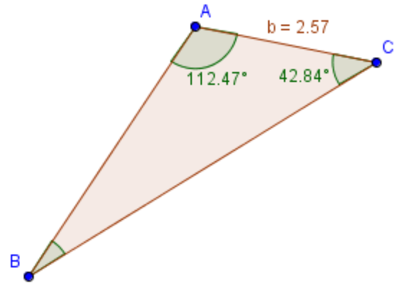
$$c' \approx 2.5$$

$\text{m}\angle B \approx 42.9°$, $\text{m}\angle C \approx 105.1°$, and $c \approx 12.8$; $\text{m}\angle B' \approx 137.1°$, $\text{m}\angle C' \approx 10.9°$, and $c' \approx 2.5$

Problems: Find the missing measurements to the nearest hundredth using the Law of Sines:

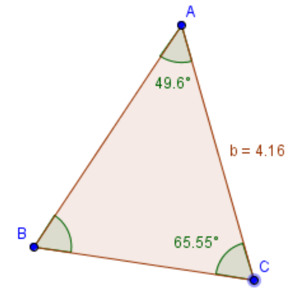
1)

$m\angle B =$
$a =$
$c =$



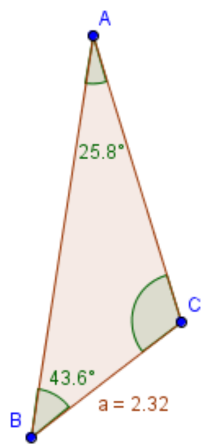
2)

$m\angle B =$
$a =$
$c =$



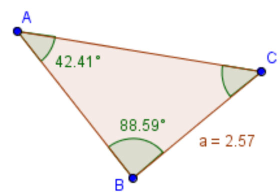
3)

$m\angle C =$
$b =$
$c =$

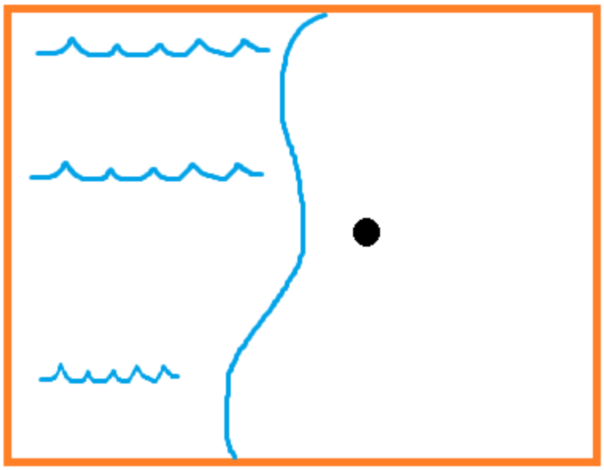


4)

$m\angle C =$
$b =$
$c =$



5) Josie, Mckenna, and Whitney take a trip to the California coastline during the summer to enjoy some time at the beach (and to work on their summer calculus homework in a more pleasant environment). Whitney wants to go for a swim, Josie fancies a nap on the beach, and Mckenna decides to study limit notation in their hotel room. Josie and Whitney walk down to the shore and Josie finds a suitable spot to doze off. Whitney runs due northwest from Josie, splashing into the water. As Whitney gets about 50 feet from Josie, Josie notices a rather large fin the in water, due west! Josie screams for Whitney to look out and points toward the fin, and Whitney looks back to Josie then turns 110 degrees clockwise and spots the fin. Whitney is frozen in fear; the perceived shark pauses, anticipating its next move.



- Draw a point in the water that represents Whitney's location.
- Connect Josie's point and Whitney's point with a line segment.
- Draw a fin in the water.
- Connect the fin to both points with line segments.
- Appropriately label the vertices and sides of the triangle.
- Add known information to the picture.

Use the Law of Sines to determine how far apart the fin and Whitney are.

6) Hearing a scream, Mckenna walks onto the patio outside of her well-built hotel room on the 8^{th} floor (approximately 80 feet above the ground). Mckenna sees the fin in the water near her classmate. The angle of depression she can view the fin with is 50 degrees. Mckenna finds a new solution to the question "when will I ever use this?" by summoning superhuman strength and hurling her calculus book from the patio, over the beach, at the base of the fin (assume the textbook travels in a straight line)!

- Sketch the triangular relationship between Mckenna's position, the fin's position, and the base of the hotel.
- Label the points and sides of the triangle. Add known information to the picture.
- Use the Law of Sines to determine how far Mckenna threw the textbook.

- Problem 6 is doable for a basic geometry student. Explain why this is the case.

7) Andrew is warming up for a racquetball match and notices a pipe in an unfinished part of the high ceiling. Having excellent aim, Andrew serves the ball at a 50 degree upward angle and the ball travels 10 meters toward the pipe. As the pipe is a curved surface, the ball ricochets back downward and travels 9 meters before it strikes the ground.

- Sketch the triangular relationship between Andrew's position, the pipe's position, and where the ball hits the ground.

- Label the points and sides of the triangle. Add known information to the picture.
- Which style of known information is given (SSA, ASA, AAS, SAS, SSS) ?
- Why does this problem have two solutions?

- How far in front of Andrew did the ball hit the ground? Use the Law of Sines to approximate both solutions.

8) Grant wants to ask his crush out on a date after school, but considers approaching her with her entire soccer team unnerving. So, his scheme is to attach an invitation to a drone, which he can fly via remote control. The drone takes off from the ground and flies at a $40°$ angle of elevation and travels 400 meters until it is directly above the field. Then, Grant triggers a release mechanism and the invitation drops down to its destination. He then flies the drone 330 meters to the ground. Assume there is no lateral movement in either flight path (i.e., from above the path of the drone is linear).

Sketch a reasonable representation of the flight of the drone.

How far from the launch site does the drone land?

9) While sharing his heroic tale of asking out a girl that was practicing soccer with a dozen other girls, one of Grant's rivals claims that he too employed a drone to accomplish the same goal. Because Grant knows that a player is never truly done improving his skills in the game, he inquires about the measurements of the rival's flight plan. The rival says that his drone left at a $55°$ angle of elevation, travelled 200 meters, and then flew another 150 meters before landing. When Grant asked whether the drone experienced any lateral movement, the rival said that there was none.

Sketch a reasonable representation of the rival's drone's flight path.

How far from the launch site does the rival's drone land?

What should Grant learn from the rival's description of the event?